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The stability of a rigidly rotating magnetic fluid column effect of a periodic azimuthal magnetic field

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Abstract. The stability of an infinitely long magnetic fluid column of weak viscous effects is investigated. The column is subjected to a periodic azimuthal magnetic field and a rigid-body rotation. Non-axisymmetric two-dimensional perturbations are considered in this investigation. Linear analysis leads to a Mathieu equation with complex coefficients. The analytical results show that the constant magnetic field plays a stabilizing role and can be used to suppress the instability due to the rotation. When the field has been oscillating, the stabilizing role of the amplitude of the magnetic field decreases somewhat due to the applied frequency ω_0. The oscillating magnetic field plays a dual role in the stability criterion. The increase of the azimuthal wavenumber decreases the unstable region due to the increase of the column radius. A small viscosity plays a destabilizing effect due to the influence of the angular velocity in the presence of a constant or an oscillating magnetic field. A magnetic column can be stabilized at a given azimuthal wavenumber by a suitable choice of the angular velocity, the density and viscosity for the outer fluid being greater than the corresponding parameters for the inside fluid.

1. Introduction

The subject of the stability of a magnetic fluid column when subjected to an applied magnetic field has attracted a great deal of attention in recent years. The stability of a magnetic fluid jet when stressed with a constant magnetic field was of considerable interest to Rosensweig [1]. It was observed experimentally that the applied magnetic field inhibits the breakup of capillary magnetic fluid jets. The stability of a magnetic fluid column was experimentally demonstrated by Zelazo and Melcher [2]. Malik and Singh [3] have investigated the stability of a magnetic fluid jet by constructing the respective Lagrangian functions and solving the associated Euler–Lagrangian equations of motion in the presence of a magnetic field when the field is either azimuthal or axial. They used the energy principle to derive the dispersion relation with their analysis based on axial perturbations.

The stability of a rigidly rotating fluid column was studied by Alterman [4]. It was found that under certain conditions the rotation may have a stabilizing or destabilizing effect. Hocking and Michael [5] demonstrated that rotation has a destabilizing effect. Bauer [6–8], in the absence of a magnetic field, has analysed a rigidly rotating fluid column in a variety of geometries. Wilson [9] investigated the effect of an axial magnetic field on the capillary instability of an infinitely long, rigidly rotating, cylindrical fluid column. A linear stability analysis has been implemented for the Taylor–Dean flow (a viscous flow between rotating concentric cylinders with a pressure gradient acting in the azimuthal direction) by Chen and

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Chang [10]. Flow between concentric cylinders with the inner cylinder rotating and an axial pressure gradient imposed in the annulus reveals a rich variety of flow regimes depending on the flow conditions. The occurrence of these flow regimes was studied experimentally by both visually and optically detecting the transition from one flow regime to another by Lueptow *et al* [11].

The phenomenon of parametric resonance arises in many branches of physics and engineering. Donnelly [12] has found experimentally that cylindrical Coutte flow can be stabilized somewhat by having the velocity of the inner wall oscillate about a mean value. The stability of a liquid jet under a time-dependent electric field has been investigated by Mohamed and Nayyar [13] and Mohamed *et al* [14]. Recently, El-Dib [15] has carried out the stability analysis of an oscillating liquid column subjected to a periodic rigid-body rotation. A Mathieu equation with a parametric imaginary damping term was obtained and analysed. El-Dib and Moatimid [16] and Moatimid and El-Dib [17] have developed theoretical analysis to investigate the effect of a periodic rotation of a cylindrical liquid jet under the influence of an axial and radial constant electric field.

The present work is to examine the stability of capillary waves of a rigidly rotating magnetic fluid column subjected to a periodic azimuthal magnetic field for non-axisymmetric two-dimensional perturbations.

2. Formulation

A magnetic fluid column performs a rigid-body rotation, in a weightless condition, with a constant angular velocity, Ω_1, about its axis of symmetry having density ρ_1 and magnetic permeability μ_1. The magnetic fluid column is embedded in a rotating unbounded magnetic fluid having density, ρ_2, magnetic permeability, μ_2, and a constant angular velocity, Ω_2. The system is subjected to an azimuthal periodic magnetic field with a forcing frequency, ω_0,

$$\underline{H} = H_0 \cos \omega_0 t \underline{e}_\theta \tag{1}$$

where $\underline{e}_\theta$ is the unit vector in the θ direction. The fluids are homogeneous, incompressible and exhibit interfacial tension. The tension forces act as restoring forces to the otherwise damped oscillations of the interfacial surfaces. In the equilibrium condition the interfacial surface exhibits a circular cylinder of radius R.

We confine the analysis to consider weak viscous effects. These effects are believed to be significant only within a thin vortical surface layer so that the motions elsewhere in the liquid column may reasonably be assumed irrotational. Thus the viscous effects are introduced through the normal damped stress term in the boundary condition at the surface of separation.

In view of the weak viscous approximation, which is considered here, the governing equations for a bulk of magnetic fluid phases are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{vu}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} \tag{3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} = -\frac{1}{\rho}\frac{\partial P}{\partial z} \tag{4}$$

where the cylindrical polar coordinates (r, θ, z) is used, P is the hydrostatic pressure and $\underline{V} = u\underline{e}_r + v\underline{e}_\theta + w\underline{k}$, represents the velocity of the liquid particle inside and outside the magnetic column. The continuity equation div $\underline{V} = 0$ takes the form

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0. \tag{5}$$

The magneto-quasistatic approximation is valid [18] for the problem at hand. With a quasistatic model, it is recognized that relevant time rates of change are sufficiently low that contributions due to a particular dynamical process are ignorable. The objective in magnetic fluids is concerned with phenomena in which magnetic energy much exceeds electrical energy storage and where the propagation times of electromagnetic waves are short compared to those of interest to us. In accordance with the validity of the quasistatic approximation a scalar function, ϕ, representing the magnetic potential, is introduced such that

$$\underline{H} = -\nabla\phi \tag{6}$$

where the differential equation satisfied by ϕ is Laplace's equation

$$\nabla^2\phi = 0. \tag{7}$$

In equilibrium $u = u^{(0)} = 0$, $v = v^{(0)} = r\Omega$ and $w = w^{(0)} = 0$. The equilibrium pressure is given by

$$P_j^{(0)} = \tfrac{1}{2}\rho_j r^2\Omega_j^2 + \mathbb{C}_j \tag{8}$$

where the superscript (0) refers to the equilibrium state and $\mathbb{C}_j$, $j = 1, 2$ are constants of integration. From the continuity of the normal stress at the interface we get the jump in the pressure to be zero, whence

$$\mathbb{C}_1 - \mathbb{C}_2 = \frac{T}{R} + \frac{1}{2}R^2(\rho_2\Omega_2^2 - \rho_1\Omega_1^2) - \frac{1}{2}(\mu_1 - \mu_2)H_0^2\cos^2\omega_0 t \tag{9}$$

where T is the surface-tension coefficient.

3. Perturbation equations and solutions

Consider the effect of a small disturbance to the interface at $r = R$. We assume that the surface deflection is given by

$$r = R + \xi(\theta, t) \tag{10}$$

where

$$\xi(\theta, t) = \gamma(t)\mathrm{e}^{\mathrm{i}m\theta} \qquad (0 \leqslant \theta \leqslant 2\pi) \tag{11}$$

$\gamma(t)$ is an unknown function of time t and the integer m is the azimuthal wavenumber.

A liquid system of infinite length exhibiting no z-dependency, which means that waves in the longitudinal direction are suppressed and the longitudinal liquid velocity $w = 0$ and $\frac{\partial}{\partial z} = 0$, is treated here. Therefore, in the two-dimensional flow case the various perturbations may be put in the form

$$F(r, \theta, t) = \hat{f}(r, t)\mathrm{e}^{\mathrm{i}m\theta} \tag{12}$$

where F stands for any (linear) physical quantity.

For two-dimensional flow the linearized form of the equations of motion may be written as

$$\rho\frac{\partial u}{\partial t}+\rho\Omega\frac{\partial u}{\partial\theta}-2\rho\Omega v=-\frac{\partial\pi}{\partial r} \tag{13}$$

$$\rho\frac{\partial v}{\partial t}+\rho\Omega\frac{\partial v}{\partial\theta}+2\rho\Omega u=-\frac{1}{r}\frac{\partial\pi}{\partial\theta} \tag{14}$$

$$\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}=0 \tag{15}$$

where the function π represents the increment of the pressure $(P=P^{(0)}+\pi)$. Introducing the stream function $\psi(r,\theta,t)$ such that

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\qquad v=\frac{\partial\psi}{\partial r}. \tag{16}$$

Then equations (13) and (14) in terms of the stream function ψ yield, after elimination of the pressure potential π, the following equation:

$$\left(r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}-m^2\right)\left(\frac{\partial}{\partial t}+\mathrm{i}m\Omega\right)\hat{\psi}(r,t)=0 \tag{17}$$

which has the solution

$$\left(\frac{\partial}{\partial t}+\mathrm{i}m\Omega\right)\hat{\psi}(r,t)=A_1^*(t)r^m+A_2^*(t)r^{-m}. \tag{18}$$

Thus the stream functions $\psi_j(r,\theta,t)$, $(j=1,2)$ inside and outside the magnetic column are

$$\psi_1(r,\theta,t)=A_1(t)r^m\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\leqslant R) \tag{19}$$

$$\psi_2(r,\theta,t)=A_2(t)r^{-m}\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\geqslant R) \tag{20}$$

where

$$A_j(t)=\left(\frac{\partial}{\partial t}+\mathrm{i}m\Omega_j\right)^{-1}A_j^*(t)\qquad j=1,2 \tag{21}$$

$A_1(t)$ and $A_2(t)$ are functions of time which are determined from the appropriate boundary conditions.

With the kinematic boundary condition

$$u_j=\frac{\partial\xi}{\partial t}+\mathrm{i}m\Omega_j\xi\qquad(r=R) \tag{22}$$

the stream functions $\psi_j(r,\theta,t)$ are

$$\psi_1(r,\theta,t)=\mathrm{i}\frac{R}{m}\left(\frac{r}{R}\right)^m\left[\frac{\mathrm{d}\gamma}{\mathrm{d}t}+\mathrm{i}m\Omega_1\gamma\right]\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\leqslant R) \tag{23}$$

$$\psi_2(r,\theta,t)=\mathrm{i}\frac{R}{m}\left(\frac{R}{r}\right)^m\left[\frac{\mathrm{d}\gamma}{\mathrm{d}t}+\mathrm{i}m\Omega_2\gamma\right]\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\geqslant R). \tag{24}$$

Also, the pressure potentials are given by

$$\pi_1=-\frac{R}{m}\rho_1\left(\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}+2\mathrm{i}(m-1)\Omega_1\frac{\mathrm{d}\gamma}{\mathrm{d}t}-m(m-2)\Omega_1^2\gamma\right)\left(\frac{r}{R}\right)^m\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\leqslant R) \tag{25}$$

$$\pi_2=\frac{R}{m}\rho_2\left(\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}+2\mathrm{i}(m+1)\Omega_2\frac{\mathrm{d}\gamma}{\mathrm{d}t}-m(m+2)\Omega_2^2\gamma\right)\left(\frac{R}{r}\right)^m\mathrm{e}^{\mathrm{i}m\theta}\qquad(r\geqslant R). \tag{26}$$

In accordance with the two-dimensional flow considered here, Laplace's equation (7) for the magnetic potential ϕ has the form

$$\left(r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}-m^2\right)\hat{\phi}(r,t)=0 \tag{27}$$

which has the solution

$$\phi_1(r,\theta,t)=B_1(t)r^m\mathrm{e}^{\mathrm{i}m\theta} \qquad (r\leqslant R) \tag{28}$$

$$\theta_2(r,\theta,t)=B_2(t)r^{-m}\mathrm{e}^{\mathrm{i}m\theta} \qquad (r\geqslant R) \tag{29}$$

where $B_1(t)$ and $B_2(t)$ are the functions of time which are determined from the appropriate boundary conditions.

The magnetic boundary conditions which have to be satisfied at the surface $r=R$ are:

(1) The magnetic potential ϕ should be continuous at the interface. Thus

$$\phi_1-\phi_2=0 \qquad (r=R). \tag{30}$$

(2) The normal component of the magnetic displacement is continuous at the interface, so that

$$\left(\mu_1\frac{\partial\phi_1}{\partial r}-\mu_2\frac{\partial\phi_2}{\partial r}\right)+\frac{1}{r}\frac{\partial\xi}{\partial\theta}(\mu_1-\mu_2)H_0\cos\omega_0 t=0 \qquad (r=R). \tag{31}$$

In view of the above conditions (30) and (31), using the solutions given by (28) and (29) the magnetic potentials inside and outside the magnetic column take the form

$$\phi_1(r,\theta,t)=\mathrm{i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left(\frac{r}{R}\right)^m\gamma(t)H_0\cos\omega_0 t\,\mathrm{e}^{\mathrm{i}m\theta} \qquad (r\leqslant R) \tag{32}$$

$$\phi_2(r,\theta,t)=\mathrm{i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left(\frac{r}{R}\right)^{-m}\gamma(t)H_0\cos\omega_0 t\,\mathrm{e}^{\mathrm{i}m\theta} \qquad (r\geqslant R). \tag{33}$$

The normal hydrodynamic stress is balanced with the normal magnetic stress [1]. Modifying the boundary condition at the surface should be an acceptable means of including the small viscous effects. Thus we have

$$\sigma_{rr}=-P+\frac{1}{2}\mu(H_r^2-H_\theta^2)+\frac{1}{2}\eta\frac{\partial u}{\partial r}. \tag{34}$$

The linearized normal stress condition is

$$\pi_2-\pi_1+R(\rho_2\Omega_2^2-\rho_1\Omega_1^2)\xi+\frac{1}{R}H_0\cos\omega_0 t\left(\mu_1\frac{\partial\phi_1}{\partial\theta}-\mu_2\frac{\partial\phi_2}{\partial\theta}\right)$$
$$+\frac{T}{R^2}(m^2-1)\xi+2\eta_1\frac{\partial u_1}{\partial r}-2\eta_2\frac{\partial u_2}{\partial r}=0 \qquad (r=R) \tag{35}$$

where η_1 and η_2 are the coefficients of viscosity inside and outside the fluid column respectively.

4. The characteristic equation

In deriving the characteristic equation we substitute from (25), (26), (32) and (33) into (35), using (11) we obtain the following dispersion equation:

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}+\frac{2}{\rho_1+\rho_2}\left\{\frac{m}{R^2}(\eta_1(m-1)+\eta_2(m+1))+\mathrm{i}(\rho_2\Omega_2(m+1)+\rho_1\Omega_1(m-1))\right\}\frac{\mathrm{d}\gamma}{\mathrm{d}t}$$
$$+\frac{m}{\rho_1+\rho_2}\left\{\frac{T(m^2-1)}{R^3}-[\rho_2\Omega_2^2(m+1)+\rho_1\Omega_1^2(m-1)]\right.$$

$$+\frac{m}{R^2}\mu^* H_0^2 \cos^2\omega_0 t + \mathrm{i}\frac{2m}{R^2}[\Omega_1\eta_1(m-1)+\Omega_2\eta_2(m+1)]\Big\}\gamma = 0 \tag{36}$$

which is a second-order differential equation with periodic coefficients of Mathieu type.

In order to eliminate the imaginary damping term from equation (36) we introduce the following transformation:

$$\gamma(t) = \Theta(t)\exp\left\{\frac{-\mathrm{i}t}{\rho_1+\rho_2}(\rho_2\Omega_2(m+1)+\rho_1\Omega_1(m-1))\right\}.$$

Then equation (36) reduces to

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} + 2\eta_0\frac{\mathrm{d}\Theta}{\mathrm{d}t} + (\delta_m + \mathrm{i}\tilde{\eta} + 4\varepsilon q\cos^2\omega_0 t)\Theta = 0 \tag{37}$$

where

$$\delta_m = \frac{1}{(\rho_1+\rho_2)}\left[m(m^2-1)\frac{T}{R^3} + \frac{1}{(\rho_1+\rho_2)}(\rho_2\Omega_2(m+1)+\rho_1\Omega_1(m-1))^2 - m(\rho_2\Omega_2^2(m+1)+\rho_1\Omega_1^2(m-1))\right] \tag{38}$$

$$\eta_0 = \frac{m}{R^2(\rho_1+\rho_2)}[\eta_1(m-1)+\eta_2(m+1)] \tag{39}$$

$$\tilde{\eta} = \frac{2m^2}{R^2(\rho_1+\rho_2)}[\Omega_1\eta_1(m-1)+\Omega_2\eta_2(m+1)]. \tag{40}$$

$$\tilde{q} = \frac{\mu^* m^2}{4R^2(\rho_1+\rho_2)}H_0^2 \qquad \mu^* = \frac{(\mu_2-\mu_1)^2}{\mu_2+\mu_1}. \tag{41}$$

The small dimensionless parameter, ε, is introduced such that $H_0^2 = \varepsilon\tilde{H}^2$ so that $\tilde{q} = \varepsilon q$.

5. Stability analysis for the case of a static magnetic field

For a static case as $\omega_0 \to 0$, equation (37) reduces to

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} + 2\eta_0\frac{\mathrm{d}\Theta}{\mathrm{d}t} + (\delta_m + 4\tilde{q} + \mathrm{i}\tilde{\eta})\Theta = 0 \tag{42}$$

which is a linear differential equation with constant coefficients and can be satisfied by $\Theta = \exp(St)$, where S is a complex constant given by

$$S^2 + 2\eta_0 S + (\delta_m + 4\tilde{q} + \mathrm{i}\tilde{\eta}) = 0. \tag{43}$$

This dispersion relation is a quadratic in S with complex coefficients. Necessary and sufficient conditions for stability can be obtained. Using the Hurwitz criterion for a quadratic polynomial with complex coefficients [19]. Hence the following conditions are imposed to control the stability criteria in the static case:

$$\eta_0 > 0 \qquad \text{and} \qquad 4\eta_0^2(\delta_m + 4\tilde{q}) - \tilde{\eta}^2 > 0. \tag{44}$$

It is noted that the first condition is trivially satisfied when $m \geqslant 1$ or $\eta_2 > \eta_1$ for all the azimuthal wavenumbers m. In the case of non-viscous fluids stability occurs when

$$\delta_m + 4\tilde{q} > 0. \tag{45}$$

In the absence of the magnetic field, the system is stable as $\delta_m > 0$.

In the absence of the angular velocity Ω_j, the above stability condition reduces to

$$(m^2-1)T + Rm\mu^* H_0^2 > 0 \tag{46}$$

and so, for the non-axisymmetric perturbations, the system is always stable. However, our results do agree with those of Malik and Singh [3]. The instability occurs if $\Omega_1 > \Omega_2$ even when the non-axisymmetric perturbations are considered.

The presence of the angular velocity, Ω_j, changes the above mechanism. In terms of the azimuthal magnetic field the stability is present when

$$H_0^2 > H_c = -\frac{R^2(\rho_1 + \rho_2)}{m^2\mu^*}\delta_m. \tag{47}$$

Condition (47) is trivially satisfied for positive values of δ_m. It can be noted that when $m = 1$, the parameter δ_m is

$$\delta_1 = \frac{2\rho_2(\rho_2 - \rho_1)\Omega_2^2}{(\rho_2 - \rho_1)^2}.$$

Thus condition (47) is trivially satisfied as $\rho_2 > \rho_1$; $m = 1$. This shows that the difference between the densities for the two fluids plays an important role in the stability criteria; a role that was not observed before. To examine the influence of the rotation on the stability, we take for simplicity $\rho_1 = \rho_2 = \rho$ in equation (38). Hence we have $\delta_m > 0$ when

$$(\Omega_2 - \Omega_1)^2 < \frac{2mT}{\rho R^3}.$$

At this stage the magnetic field has no effect. While the stabilizing influence of the magnetic field appears when

$$(\Omega_2 - \Omega_1)^2 > \frac{2mT}{\rho R^3}.$$

For negative values of δ_m the stability condition (45) can be satisfied if the inequality (46) is true. A case which was not possible in the absence of the magnetic field. This shows the stabilizing nature of the constant azimuthal magnetic field. The increase of the angular velocity, Ω_j, increases the term δ_m and large values of H_0^2 are needed to suppress the destabilizing effect due to the rotation.

In the presence of viscosity, the critical magnetic field, H_c, becomes:

$$H_c^* = -\frac{R^2(\rho_1 + \rho_2)}{m^2\mu^*}\delta_m^* \tag{48}$$

where

$$\delta_m^* = \delta_m - \tilde{\eta}^2/4\eta_0^2. \tag{49}$$

The stability is trivially satisfied as $\delta_m > \tilde{\eta}^2/4\eta_0^2$. Otherwise large values of H_0^2 are needed to suppress the destabilizing effect due to the presence of a small viscosity in a rotating medium.

5.1. Numerical estimation in the case of a static field

Due to the importance of the sign of the dispersion term δ_m, in the absence or the presence of the magnetic field, the numerical calculations for δ_m are made for a sample example.

The plane (δ_m, R) shown in figures 1 and 2 depicts the dispersion term δ_m from relation (38) for a particular system where $\rho_1 > \rho_2$. The graph contains the curves $\delta_2 : \delta_6$. The stable and unstable regions are denoted by the symbols S and U respectively. The case of $\Omega_1 < \Omega_2$ appears in figure 1, while the case of $\Omega_1 > \Omega_2$ is shown in figure 2. Instability is revealed in the case of $m = 1$ as $\rho_1 > \rho_2$. For $m > 1$, the stability occurs for relatively small values of the radius R. Suddenly instability occurs as R is increased. The increase in

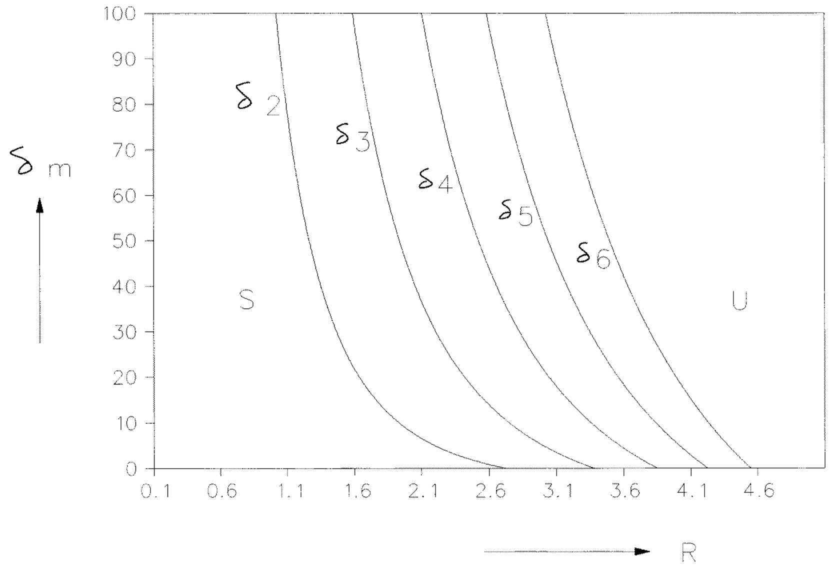

Figure 1. The dispersion term δ_m for a system having $\rho_1 = 0.998\,23$ g cm^{-3}, $\rho_2 = 0.879$ g cm^{-3}, $T = 35$ dynes cm^{-1}, $\Omega_1 = 3$ and $\Omega_2 = 5$

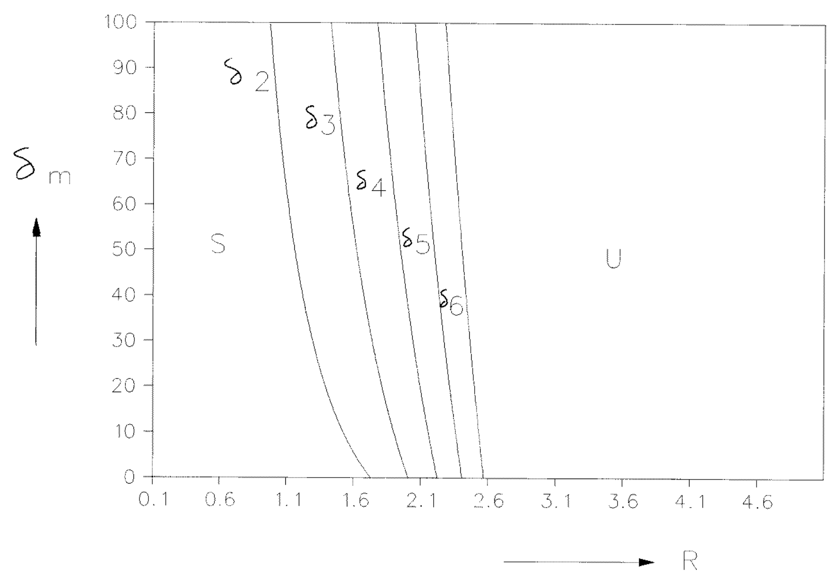

Figure 2. As in figure 1, except that $\Omega_1 > \Omega_2$ ($\Omega_1 = 8$ and $\Omega_2 = 3$).

the azimuthal wavenumber increases the stable region. It is observed that, as $\Omega_1 > \Omega_2$, the stable region has decreased to leave a large unstable region. This shows that a destabilizing influence arises as the fluid column rotates faster than the fluid in the outer column.

In figure 3, we plot $\log(H_0^2)$ versus the radius R. The graph depicts the critical magnetic curve, H_c, from relation (47) which is separating the stable region from the unstable region. Six different azimuthal wavenumbers ($m = 1 : 6$) are presented. The region which lies

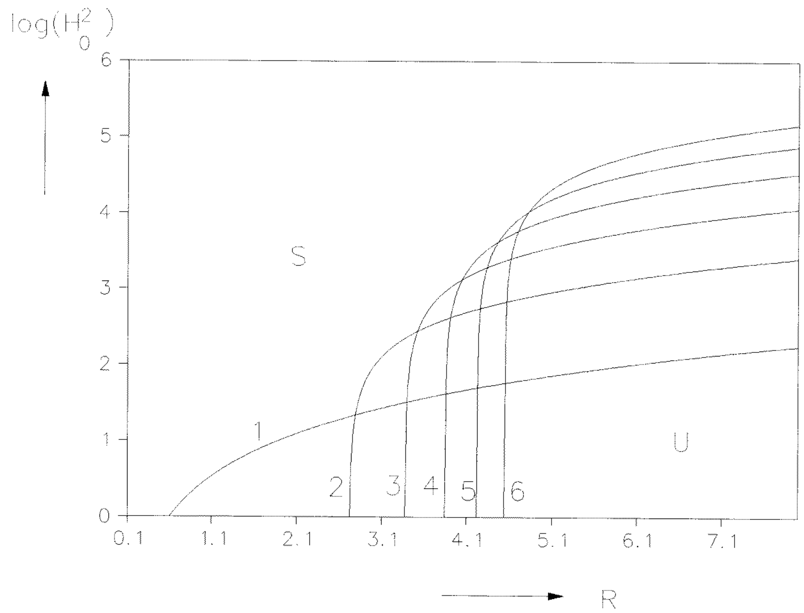

Figure 3. The ($\log H_0^2$, R)-plane for the system as in figure 1. The curves are given by the relation (47) for $m = 1:6$ and marked by the numbers 1–6 respectively.

above the curve H_c is the stable region. The region which lies below the critical curve H_c denotes the unstable region. It is found that the increase of the azimuthal wavenumber decreases the stable region as H_0 is increased and decreases the unstable region as R is increased. We observe that as R is increased, larger values of H_0^2 are required in order to achieve the stability. The stable region for specified m is increased as the constant azimuthal magnetic field is increased. Thus the magnetic field plays a stabilizing role.

In figure 4, the graph is a comparison between the system of non-viscous fluids with that system having a small viscosity. The calculations for the transition curves, H_c and H_c^*, are made in the case of $m = 1$ as an example. Due to the presence of the viscosity the unstable region is increased. It is clear from equation (49) that, as Ω_j is increased, the parameter $\tilde{\eta}$ is increased and it follows that larger values of H_0^2 are needed to achieve the stability. This shows that the rotating fluids in the presence of a small viscosity lead to the existence of an additional unstable region that was not found in the rotating fluids of a non-viscous column. It can be noted that this additional unstable region will disappear in the case of non-rotating fluids even in the presence of viscosity.

6. Non-viscous column stressed by an oscillating field

As a limiting case, where η_1 and η_2 tend to zero, equation (37) reduces to

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} + (\delta_m + 4\tilde{q}\cos^2\omega_0 t)\Theta = 0. \tag{50}$$

In order to put equation (50) in the canonical form, we use the following notations:

$$a = \frac{1}{\omega_0^2}(\delta_m + 2\tilde{q}) \qquad q_0 = \frac{\tilde{q}}{\omega_0^2} \qquad \text{and} \qquad \tau = \omega_0 t.$$

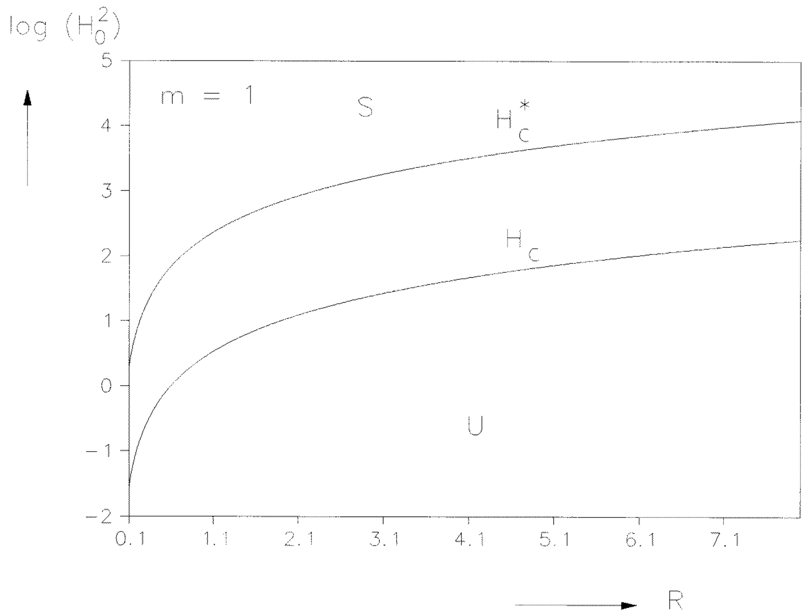

Figure 4. The $(\log H_0^2,\ R)$-plane for a constant magnetic field. Two transition curves are displayed, H_c^* (for the viscous case) and H_c (for the non-viscous case).

Thus we have

$$\frac{d^2\Theta}{d\tau^2} + (a + 2q_0 \cos 2\tau)\Theta = 0. \tag{51}$$

Equation (51) is the well-known Mathieu differential equation which has been studied extensively.

The condition for stability reduces to the problem of the bounded regions of the Mathieu functions for which Mclachlan [20] gives the condition of stability as

$$\tilde{q}^2 + 4(\omega_0^2 - \delta_m)\tilde{q} + 2\delta_m(\omega_0^2 - \delta_m) > 0. \tag{52}$$

For arbitrary frequency ω_0 condition (52) can be satisfied when

$$\delta_m + 2\tilde{q} > 0 \tag{53}$$

$$\tilde{q}^2 - 4\delta_m\tilde{q} - 2\delta_m^2 > 0. \tag{54}$$

In terms of the amplitude of the magnetic field H_0 conditions (53) and (54) become

$$H_0^2 > 2H_c \tag{55}$$

and

$$H_0^4 + 16H_cH_0^2 - 32H_c^2 > 0 \tag{56}$$

namely

$$(H_0^2 + 4(2+\sqrt{6})H_c)(H_0^2 + 4(2-\sqrt{6})H_c) > 0. \tag{57}$$

In view of the sign of δ_m, condition (57) reduces to

$$H_0^2 + 4(2+\sqrt{6})H_c > 0 \qquad \delta_m > 0$$
$$H_0^2 + 4(2-\sqrt{6})H_c > 0 \qquad \delta_m < 0.$$

Since condition (55) is trivially satisfied when $\delta_m < 0$, and $4(\sqrt{6}-2) < 2$, the stability occurs due to the presence of the frequency ω_0 as

$$H_0^2 + 4(2+\sqrt{6})H_c > 0 \qquad \delta_m > 0 \tag{58}$$

and

$$H_0^2 > 2H_c \qquad \delta_m < 0. \tag{59}$$

The comparison between the stability condition in the case of a static field ($H_0^2 > H_c$) and the stability conditions mentioned above shows that the oscillating magnetic field plays a stabilizing role. But the field frequency ω_0 has contracted this influence.

In the terms of H_0^2 the stability condition (52) can be rearranged in the form:

$$H_0^4 + 16\frac{R^2(\rho_1+\rho_2)}{m^2\mu^*}(\omega_0^2-\delta_m)H_0^2 + 32\frac{R^4(\rho_1+\rho_2)^2}{m^4\mu^{*2}}\delta_m(\omega_0^2-\delta_m) > 0 \tag{60}$$

which can be written in the form

$$(H_0^2 - H_1^*)(H_0^2 - H_2^*) > 0. \tag{61}$$

It follows that the above condition is satisfied if

$$H_0^2 > H_1^* \qquad \text{or} \qquad H_0^2 < H_2^* \qquad (H_1^* > H_2^*)$$

where

$$H_{1,2}^* = \frac{8R^2(\rho_1+\rho_2)}{m^2\mu^*}[-(\omega_0^2-\delta_m) \pm ((\omega_0^2-\delta_m)(\omega_0^2-\tfrac{3}{2}\delta_m))^{1/2}]. \tag{62}$$

For arbitrary values of H_0, condition (60) can be satisfied when both H_1^* and H_2^* have negative values. This can be achieved by using the Hurwitz examination. Hence, two conditions are required:

$$\omega_0^2 - \delta_m > 0 \qquad \text{and} \qquad \delta_m > 0.$$

The first condition is trivially satisfied for negative δ_m, while the second condition falls. This means that a stable region or an unstable region does not reveal for all values of H_0. Thus a stable region appears here that was unstable in the absence of the field. Since the system is stable in the absence of the field as $\delta_m > 0$, instability occurs for all values of H_0 as the field frequency ω_0 satisfies $\omega_0^2 < \delta_m$; $\delta_m > 0$.

In graphing condition (60), we are careful to display a broken curve that represents the case of $\delta_m = 0$ shown in figure 5. This broken curve separates the stable region ($\delta_m > 0$) from the unstable region ($\delta_m < 0$) in the absence of the field. The curve H_1^* is calculated from relation (62) which represents the transition curve where H_2^* has negative values for all R. The stability diagram in the presence of an oscillating magnetic field shows that there is a stable region, S^*, that was unstable when $H_0 = 0$. Also, there is an unstable region, U^*, that was stable in the absence of the field.

Inspection of figures 3 and 5 shows that the presence of the frequency ω_0 makes the field more stabilizing in the case of $\delta_m < 0$ and the appearance of the unstable region U^* in the case of $\delta_m > 0$ does not appear in figure 3, for the case of a constant field. This shows that the presence of the field frequency plays a dual role in the stability criteria. Figure 6, represents the stability diagram as given in figure 5 for $m = 1, 2, 3, 4$. It is found that the increase of the azimuthal wavenumber m increases both the stable region S^* and the unstable region U^*.

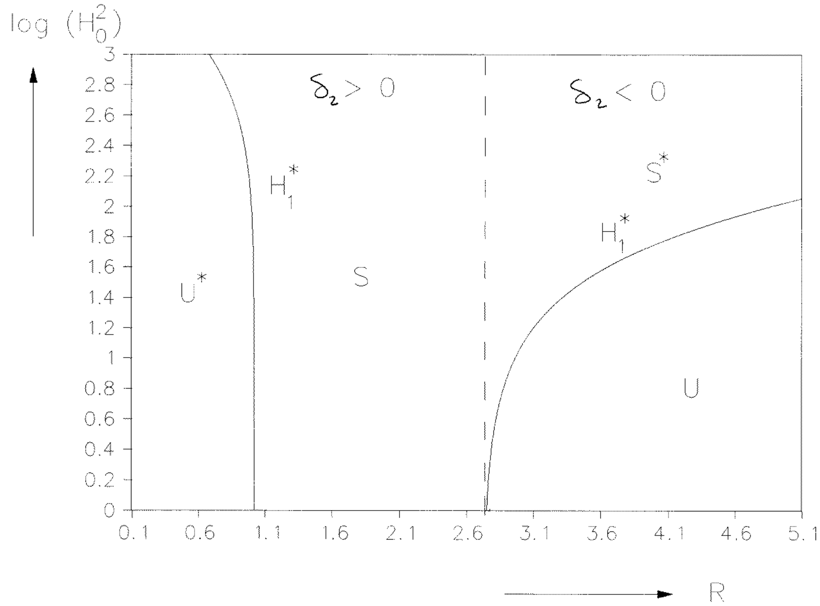

Figure 5. The same stability diagram considered in figure 4 for an oscillating magnetic field with frequency $\omega_0 = 10$ Hz. The calculations are made for the transition curves (62). The broken curve shows the case of $\delta_2 = 0$.

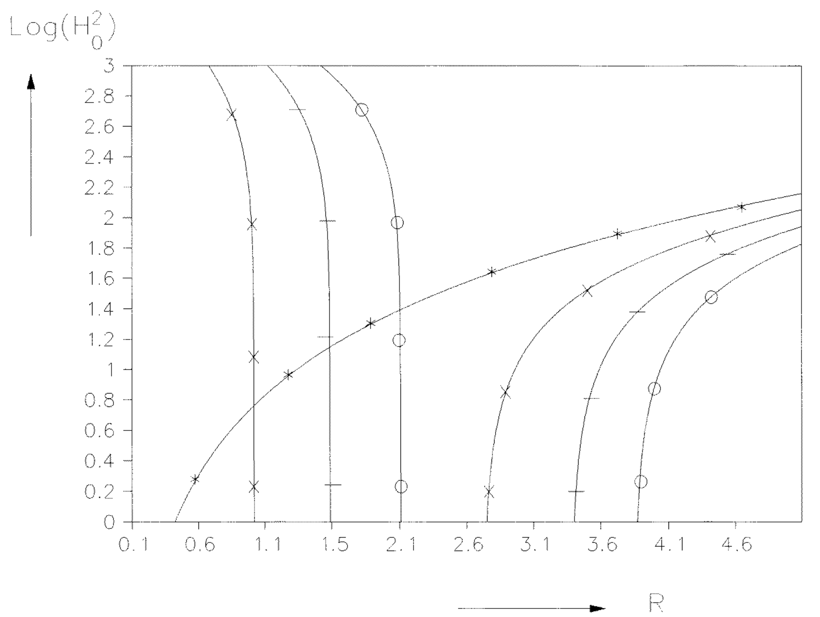

Figure 6. For the same system considered in figure 5 with the variation of the wavenumber m ($m = 1:4$). $*$ refers to the case of $m = 1$, $\times$ refers to the case of $m = 2$, $-$ represents the case of $m = 3$ and $\bigcirc$ refers to the case of $m = 4$.

7. Viscous damping effects and multiple time-scales formulation

An equation of the form (37) is known as the damped Mathieu equation with complex coefficients. We now need to determine the structure of the stability conditions for the damped Mathieu equation (37). To accomplish this, we may use a perturbation technique.

We use the method of multiple scales as described by Nayfeh and Mook [21] to obtain an approximate solution and analyse the stability criteria. In accordance with this scheme a fast time-scale, T_0, and slow time-scale, T_1, are introduced such that $T_0 = t$ and $T_1 = \varepsilon t$. The differential operator can now be expressed as the derivative expansions:

$$\frac{\mathrm{d}\,..}{\mathrm{d}t} = \frac{\partial\,..}{\partial T_0} + \varepsilon\frac{\partial\,..}{\partial T_1} + \varepsilon^2\frac{\partial\,..}{\partial T_2} + \cdots$$
$$\frac{\mathrm{d}^2\,..}{\partial t^2} = \frac{\partial^2\,..}{\partial T_0^2} + 2\varepsilon\frac{\partial^2\,..}{\partial T_0\partial T_1} + \varepsilon^2\left(\frac{\partial^2\,..}{\partial T_1^2} + 2\frac{\partial^2\,..}{\partial T_0\partial T_2}\right) + \cdots.$$

One assumes that the solution of equation (37) can be represented by

$$\Theta(t;\varepsilon) = \Theta_0(T_0,T_1) + \varepsilon\Theta_1(T_0,T_1) + \varepsilon^2\Theta_2(T_0,T_1) + \cdots. \tag{63}$$

We insert the perturbed solution (63) in Mathieu's equation (37), transform the time derivatives and collect coefficients of each power of ε. These equations must hold independently because powers of ε are linearly independent. The resulting equations can be solved successively. Thus we have

$$\varepsilon^0:\ \frac{\partial^2\Theta_0}{\partial T_0^2} + 2\eta_0\frac{\partial\Theta_0}{\partial T_0} + (\delta_m + \mathrm{i}\tilde{\eta})\Theta_0 = 0 \tag{64}$$

$$\varepsilon^1:\ \frac{\partial^2\Theta_1}{\partial T_0^2} + 2\eta_0\frac{\partial\Theta_1}{\partial T_0} + (\delta_m + \mathrm{i}\tilde{\eta})\Theta_1 = -2\frac{\partial^2\Theta_0}{\partial T_0\partial T_1} - 2\eta_0\frac{\partial\Theta_0}{\partial T_1} - 4q\Theta_0\cos^2\omega_0 T_0. \tag{65}$$

With this approach it turns out to be convenient to write the solution of equation (64) in the form

$$\Theta_0(T_0,T_1) = \mathbb{A}(T_1)\exp[(\sigma + \mathrm{i}\omega)T_0] + CC \tag{66}$$

where $\mathbb{A}$ is an unknown complex function, CC represents a complex conjugate, σ and ω are real. Both σ and ω are satisfied in the following equations:

$$\sigma^2 - \omega^2 + 2\sigma\eta_0 + \delta_m = 0 \qquad \text{and} \qquad 2\sigma\omega + 2\omega\eta_0 + \tilde{\eta} = 0. \tag{67}$$

The elimination of the parameter σ yields

$$4\omega^4 + 4(\eta_0^2 - \delta_m)\omega^2 - \tilde{\eta}^2 = 0. \tag{68}$$

This dispersion relation is a quadratic in ω^2 with complex coefficients and having two different roots. We will consider only the positive root for convenience. Necessary and sufficient conditions for stability in the zero-order case can be obtained from condition (44) as $\tilde{q}$ tends to zero. Hence the following conditions are imposed to control the stability criteria in the zero-order case:

$$\eta_0 > 0 \qquad \text{and} \qquad \delta_m > \tilde{\eta}^2/4\eta_0^2. \tag{69}$$

The solution of equation (65) (to the first order in ε) can be obtained by the knowledge of the zero-order solution in ε. Substituting equation (66) into (65) yields

$$\begin{aligned}\frac{\partial^2\Theta_1}{\partial T_0^2} + 2\eta_0\frac{\partial\Theta_1}{\partial T_0} + (\delta_m + \mathrm{i}\tilde{\eta})\Theta_1 = -2\left\{[(\sigma+\eta_0) + \mathrm{i}\omega]\frac{\partial\mathbb{A}}{\partial T_1} + q\mathbb{A}\right\}\exp[(\sigma+\mathrm{i}\omega)T_0] \\ -q\mathbb{A}\{\exp[\mathrm{i}(\omega+2\omega_0)T_0] + \exp[\mathrm{i}(\omega-2\omega_0)T_0]\}\exp(\sigma T_0) + CC.\end{aligned} \tag{70}$$

Equation (70) contains non-homogeneous terms. The uniform solution is required to eliminate the secular terms. This elimination introduces the solvability condition corresponding to the terms containing the factor $\exp[(\sigma + \mathrm{i}\omega)T_0]$. Thus, in order to analyse the solution of equation (70) we need to distinguish between two cases: the first one is the non-resonance case, when the frequency ω_0 of the oscillating magnetic field is not near the frequency ω, the second case is the resonance case which arises when the frequency ω_0 is near ω.

(i) The case when ω_0 is not near ω (the non-resonant case). In order to obtain a uniformly valid expansion the coefficient of the factor $\exp[(\sigma + \mathrm{i}\omega)T_0]$ in equation (70) must vanish. Thus we have

$$\frac{\partial \mathbb{A}}{\partial T_1} - \frac{q(\tilde{\eta} + 2\mathrm{i}\omega^2)}{2\omega(2\omega^2 + \eta_0^2 - \delta_m)}\mathbb{A} = 0 \tag{71}$$

where equations (67) are used. The solution of equation (71) shows that stability occurs in the non-resonant case when $\tilde{\eta}\omega < 0$.

(ii) The resonant case. In order to obtain a solution in the neighbourhood of the resonant case, we express the nearness of ω_0 to ω by introducing the detuning parameter ζ according to

$$\omega_0 = \omega + \varepsilon\zeta, \tag{72}$$

and hence

$$-\mathrm{i}(\omega - 2\omega_0)T_0 = \mathrm{i}\omega T_0 + 2\mathrm{i}\zeta T_1. \tag{73}$$

Thus the secular terms can be eliminated when

$$\frac{\partial \mathbb{A}}{\partial T_1} + \frac{q\omega}{2\mathrm{i}\omega^2 - \tilde{\eta}}\{2\mathbb{A} + \bar{\mathbb{A}}\exp(2\mathrm{i}\zeta T_1)\} = 0. \tag{74}$$

This equation admits a non-trivial solution of the form

$$\mathbb{A}(T_1) = (\alpha(T_1) + \mathrm{i}\beta(T_1))\exp(\mathrm{i}\zeta T_1) \tag{75}$$

with real functions α and β. Substituting (75) into equation (74) and separating the solvability condition into real and imaginary parts we obtain the equations governing α and β in the form

$$\left(\frac{\partial}{\partial T_1} - \frac{3\tilde{\eta}q}{4\omega(2\omega^2 + \eta_0^2 - \delta_m)}\right)\alpha(T_1) - \left(\zeta - \frac{q\omega}{2(2\omega^2 + \eta_0^2 - \delta_m)}\right)\beta(T_1) = 0 \tag{76}$$

$$\left(\frac{\partial}{\partial T_1} - \frac{\tilde{\eta}q}{4\omega(2\omega^2 + \eta_0^2 - \delta_m)}\right)\beta(T_1) + \left(\zeta - \frac{3q\omega}{2(2\omega^2 + \eta_0^2 - \delta_m)}\right)\alpha(T_1) = 0. \tag{77}$$

These coupled linear equations have the solutions:

$$\alpha(T_1) = \left(\zeta - \frac{q\omega}{2(2\omega^2 + \eta_0^2 - \delta_m)}\right)\exp(QT_1) \tag{78}$$

$$\beta(T_1) = \left(Q - \frac{3\tilde{\eta}q}{4\omega(2\omega^2 + \eta_0^2 - \delta_m)}\right)\exp(QT_1) \tag{79}$$

where the constant, Q, is given by

$$Q^2 - \left(\frac{\tilde{\eta}q}{\omega(2\omega^2 + \eta_0^2 - \delta_m)}\right)Q + \left\{\frac{3\tilde{\eta}^2q^2}{16\omega(2\omega^2 + \eta_0^2 - \delta_m)} + \left(\zeta - \frac{3q\omega}{2(2\omega^2 + \eta_0^2 - \delta_m)}\right)\right.$$
$$\left.\times\left(\zeta - \frac{q\omega}{2(2\omega^2 + \eta_0^2 - \delta_m)}\right)\right\} = 0. \tag{80}$$

Dispersion relation (80) is a quadratic equation in the growth rate Q. Necessary and sufficient conditions for stability are governed by the following inequalities: $\omega\tilde{\eta} < 0$ and

$$\zeta^2 - \left(\frac{2q\omega}{(2\omega^2 + \eta_0^2 - \delta_m)}\right)\zeta - \frac{3q^2}{4(2\omega^2 + \eta_0^2 - \delta_m)} > 0. \tag{81}$$

This inequality can be satisfied when

$$(\zeta - \zeta_1^*)(\zeta - \zeta_2^*) > 0$$

i.e.

$$\zeta > \zeta_1^* \qquad \text{and} \qquad \zeta < \zeta_2^* \qquad (\zeta_1^* > \zeta_2^*) \tag{82}$$

where

$$\zeta_{1,2}^* = \frac{q}{2(2\omega^2 + \eta_0^2 - \delta_m)}\{2\omega \pm \sqrt{3\delta_m - 2\omega^2 - 3\eta_0^2}\}. \tag{83}$$

In view of (72) and in terms of the amplitude of the magnetic field H_0^2 the stability conditions in the resonance case can be sought in the form

$$H_0^2 < H_1^{**} = \frac{8R^2(\rho_1 + \rho_2)(\omega_0 - \omega)(2\omega^2 + \eta_0^2 - \delta_m)}{m^2\mu^*\left(2\omega + \sqrt{3\delta_m - 2\omega^2 - 3\eta_0^2}\right)} \tag{84}$$

$$H_0^2 > H_2^{**} = \frac{8R^2(\rho_1 + \rho_2)(\omega_0 - \omega)(2\omega^2 + \eta_0^2 - \delta_m)}{m^2\mu^*\left(2\omega - \sqrt{3\delta_m - 2\omega^2 - 3\eta_0^2}\right)} \tag{85}$$

provided that $\delta_m^* > 0$. The curves H_1^{**} and H_2^{**} represent the transition curves separating stable from unstable regions. In the limiting case of a non-viscous fluid we have

$$\lim_{\eta_j \to 0} \omega^2 = \delta_m$$

hence the above transition curves reduce to

$$\hat{H}_1 = \frac{8R^2(\rho_1 + \rho_2)(\omega_0 - \sqrt{\delta_m})\sqrt{\delta_m}}{3m^2\mu^*} \tag{86}$$

$$\hat{H}_2 = \frac{8R^2(\rho_1 + \rho_2)(\omega_0 - \sqrt{\delta_m})\sqrt{\delta_m}}{m^2\mu^*}. \tag{87}$$

7.1. A numerical illustration

The numerical calculations for the transition curves H_1^{**} and H_2^{**} (for the viscous case) and the transition curves $\hat{H}_1$ and $\hat{H}_2$ (for the non-viscous case) in the resonance case of ω_0 is near ω are displayed in figure 7. In this graph H_0^2 is plotted versus the column radius R. The calculations are made in the case for both $\delta_m > 0$ and $\delta_m^* > 0$, for a system having $\rho_1 = 0.879$ g cm^{-3}, $\rho_2 = 0.998\,23$ g cm^{-3}, $\Omega_1 = 8$ and $\Omega_2 = 3$, $\omega_0 = 20$ Hz, $\eta_1 = 0.8$ g cm^{-1} s^{-1} and $\eta_2 = 1.5$ g cm^{-1} s^{-1} and $\mu^* = 37$ with the azimuthal wavenumber $m = 2$.

The curves labelled $\times$ represent the transition curves in the non-viscous case. The curves labelled $\circ$ refer to the transition curves in the presence of a small viscosity. The broken curve represents the case of $\delta_m^* = 0$. This line separates the zero-order stable region (S region) from the zero-order unstable region (U region). The location of the broken curve depends on the viscosity η_j, the angular velocity Ω_j and the azimuthal wavenumber m.

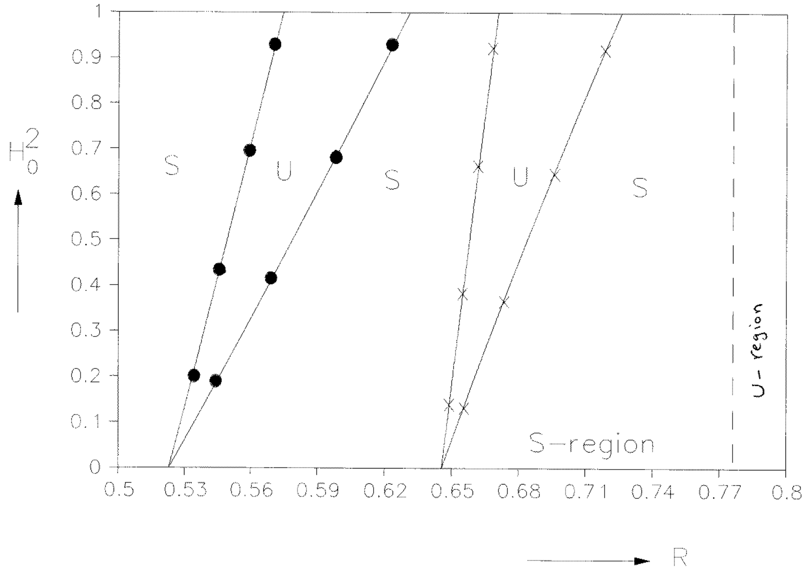

Figure 7. The stability diagram in the resonance case of ω_0 is near ω. For a system having $\rho_1 = 0.879$ g cm^{-3}, $\rho_2 = 0.998\,23$ g cm^{-3}, $\Omega_1 = 8$ and $\Omega_2 = 3$, $\omega_0 = 20$ Hz, $\eta_1 = 0.8$ g cm^{-1} s^{-1} and $\eta_2 = 1.5$ g cm^{-1} s^{-1} and $\mu^* = 37$ with the azimuthal wavenumber $m = 2$. The curves labelled $\times$ refer to the non-viscous case, while curves labelled ● refer to the viscous case. The broken curve represents the case of $\delta_2^* = 0$. The diagram indicates the transition curves (84)–(87).

It is observed that the S region is decreased as the parameter $\tilde{\eta}$ is increased. In the stability diagram shown by figure 7 the broken curve lies at $R = 0.773\,845$, the non-viscous resonance point lies at $R = 0.645\,24$ while the resonance point for the viscous transition curves is at $R = 0.523\,103$. The graph shows that the resonance region has slightly increased in the width due to the presence of a small viscosity. Also, the resonance point has been shifted to the direction of decreasing the radius R.

Figure 8 represents the same system as in figure 7, but in the viscous case where η_1 and η_2 have interchanged ($\eta_1 = 1.5$ g cm^{-1} s^{-1} and $\eta_2 = 0.8$ g cm^{-1} s^{-1}), with two cases for the field frequency ω_0 ($\omega_0 = 20$ Hz and $\omega_0 = 25$ Hz), while the other parameters are held fixed. It is found that as $\eta_1 > \eta_2$ the S region has decreased in the width with an increase in the U region, where the broken curve lies at $R = 0.652\,025$. Furthermore the resonance point has shifted to the direction of increasing R and lies at $R = 0.600\,616$, where the frequency $\omega_0 = 20$ Hz. When the frequency ω_0 is changed to the value 25 Hz, the resonance region has slightly decreased with large shifting for the resonance point to the value $R = 0.506\,101$. Thus one can say that the resonance point was affected by the field frequency ω_0 and the azimuthal wavenumber m. The increase of the frequency ω_0 decreases the value of the resonance point, while the increase of m increases this resonance point.

8. Conclusion

In this work a rigid-body rotating magnetic fluid column of a weak viscosity is formulated in the presence of a periodic azimuthal magnetic field. With a weak viscosity, it is

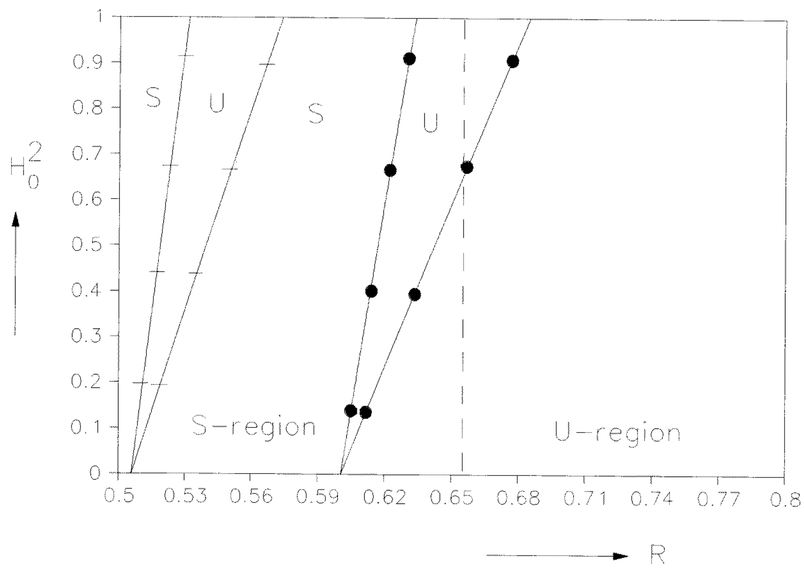

Figure 8. The same system as in figure 7, except that $\eta_1 = 1.5$ g cm^{-1} s^{-1} and $\eta_2 = 0.8$ g cm^{-1} s^{-1}, with two different values of ω_0 ($\omega_0 = 20$ Hz and $\omega_0 = 25$ Hz). − refers to the case $\omega_0 = 25$ Hz. ● refers to the case $\omega_0 = 20$ Hz.

recognized that viscous effects are included in the boundary condition of the normal stress tensor balance. So that the field equation governing the fluid motion is the Laplace equation.

Accordingly the linear analysis, a parametric dispersion equation of Mathieu type, is obtained with complex coefficients. The perturbation analysis of the multiple scales method is used. A parametric resonance occurs as the field frequency ω_0 approaches the perturbation frequency ω.

The stability examination yields the following results.

(i) Small values of the radius R has a stabilizing effect. The destabilizing influence suddenly appears as R is increased. The presence of a constant magnetic field suppresses the destabilizing role for large values of R. A more stabilizing influence for the radius R occurs as $\Omega_2 > \Omega_1$ or $\rho_2 > \rho_1$ or $\eta_2 > \eta_1$. The stability reveals for all values of R in the case of $m = 1$ and $m = 2$ as $\rho_2 > \rho_1$.

(ii) The increase of the azimuthal wavenumber m increases the stable region in the absence of the magnetic field. In the presence of a constant magnetic field, a more stabilizing influence for the radius R is associated with contraction for the stabilizing influence of the field as m is increased.

(iii) A small viscosity plays a destabilizing influence due to the presence of the angular velocity Ω_j. This destabilizing effect still occurs in the presence of the constant or the periodic magnetic field.

(iv) A dual role for the field frequency ω_0 is observed where some unstable region is changed to a stable region associated whilst some stable region is changed to an unstable region in the presence of an oscillating magnetic field.

References

[1] Rosensweig R E 1985 *Ferrodynamics* (Cambridge: Cambridge University Press)
[2] Zelazo R E and Melcher J R 1969 *J. Fluid Mech.* **39** 1
[3] Malik S K and Singh M 1991 *Int. J. Engng Sci.* **29** 1493
[4] Alterman Z 1961 *Phys. Fluids* **4** 955
[5] Hocking L M and Michael D H 1959 *Mathematika* **6** 25
[6] Bauer H F 1983 *Forschr. Ing.-Wes.* **49** 117
[7] Bauer H F 1984 *ZAMM* **64** 475
[8] Bauer H F 1989 *Flugwiss. Welt.* **13** 248
[9] Wilson S K 1992 *Q. J. Mech. Appl. Math.* **45** 363
[10] Chen F and Chang M H 1992 *J. Fluid Mech.* **243** 443
[11] Lueptow R M, Docter A and Min K 1992 *Phys. Fluids* **4** 2446
[12] Donnelly R J 1969 *Proc. R. Soc.* A **312** 130
[13] Mohamed A A and Nayyar N K 1970 *J. Phys. A: Math. Gen.* **3** 296
[14] Mohamed A A, El-Sakka A G and Sultan G M 1985 *Phys. Scr.* **31** 193
[15] El-Dib Y O 1996 *Fluid Dyn. Res.* **18** 17
[16] El-Dib Y O and Moatimid G M 1994 *Physica* **205A** 511
[17] Moatimid G M and El-Dib Y O 1994 *Int. J. Engng Sci.* **32** 1183
[18] Melcher J R 1981 *Continuum Electromechanics* (Cambridge, MA: MIT)
[19] Zahreddine Z and Elshehawey E F 1988 *Ind. J. Pure Appl. Math.* **19** 963
[20] Mclachlan N W 1964 *Theory and Application of Mathieu Functions* (New York: Dover)
[21] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillation* (New York: Wiley)